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# Nonlinear Landau Damping of Oscillations in a Bounded Plasma

DAVID MONTGOMERY

*Institute for Fluid Dynamics and Applied Mathematics, University of Maryland,* ~~Maryland~~ Maryland U., *College Park,* ~~Maryland~~

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A previously developed perturbation-theory-to-all-orders formalism is applied to the oscillations of a "collisionless" electron plasma which is bounded by perfectly reflecting walls. The long-time damping rate is the same for the $n$th order electric field as for the first order. This result generally does *not* apply to the unbounded plasma.

(Author reprint)

## I. INTRODUCTION

It has recently proved possible to give a full solution[1] to the problem of the linearized motions of a "collisionless" electron plasma which is confined by perfectly reflecting walls. The walls could be infinite parallel plates, for the one-dimensional case, or a rectangular parallelipiped. The device was similar to the method of images; it was possible to find a uniquely defined unbounded situation which reduced to the desired result within the boundaries, and which automatically satisfied the reflection conditions for all time and all velocities. The techniques of Landau[2] could then be applied to the equivalent unbounded situation.

It is also possible, as was shown some time ago,[3,4] to give a perturbation-theory-to-all-orders solution to the problem of the nonlinear oscillations of the unbounded electron plasma. The main result of reference 4 was to show that the phenomenon of Landau damping, if present in first order (the linear Landau theory), will persist to all orders.

A natural question to ask, and one which could not be answered previously, is: How does the $n$th-order damping rate compare with the first-order rate? This question is very involved for the unbounded case, but becomes almost trivial for the bounded situation, by virtue of the lower bound on absolute value of allowed wavenumber which is introduced by the walls. The result, as will be seen below, is that the damping of the $n$th order charge density (*not* distribution function) goes at the same rate as for the first order for long times.

In Sec. II, the contents of reference 4 are summarized, and what we hope is a more lucid and graphic demonstration of the principal result of reference 4 is provided. The previously stated result for the $n$th order damping rate is proved in Sec. III. Section IV discusses the result, and also contains some comments on an alternative procedure which has recently been put forward.

## II. THE PERSISTENCE OF DAMPING

We restrict ourselves for simplicity to a one-dimensional system. A "collisionless" electron (charge $-e$, mass $m$) plasma is assumed to move through a uniform immobile background of positive charge of density $eN_0$. The electron distribution function

[1] D. Montgomery and D. Gorman, Phys. Fluids **5**, 947 (1962). See also S. Gartenhaus, Phys. Fluids **6**, 451 (1963).
[2] L. D. Landau, J. Phys. (USSR) **10**, 25 (1946).
[3] D. Montgomery, Phys. Rev. **123**, 1077 (1961).
[4] D. Montgomery and D. Gorman, Phys. Rev. **124**, 1309 (1961) [Erratum, Phys. Rev. **126**, 2261 (1962)].

$f(x, v, t)$ is assumed to obey the Boltzmann–Vlasov equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = 0, \tag{1a}$$

where

$$\frac{\partial E}{\partial x} = 4\pi e\left(N_0 - \int_{-\infty}^{\infty} f\, dv\right), \tag{1b}$$

given $f(x, v, 0)$ for all $x$, $v$.

We seek a formal solution of the type

$$E = \sum_1^\infty E^{(n)}; \qquad f = \sum_0^\infty f^{(n)}, \tag{2}$$

where terms with the same superscript are assumed to be of the same order in the amplitude. It is assumed that of the various $f^{(n)}$, only $f^{(0)}$ and $f^{(1)}$ are nonvanishing at $t = 0$. If $f^{(0)} \equiv f_0(v)$ is taken to be the spatially homogeneous part of $f(x, v, 0)$, then the $n = 1$ terms of the series (2) are identical with the Landau result.[2] If we demand that the disturbance $f^{(1)}(x, v, 0)$ introduce no net charge into the system, we have

$$\int f_0(v)\, dv = N_0; \qquad \iint f^{(1)}(x, v, 0)\, dx\, dv = 0. \tag{3}$$

If Fourier–Laplace transforms are taken,

$$f_{kp}^{(n)} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-ikx} \int_0^\infty dt\, e^{-pt} f^{(n)}(x, v, t), \tag{4}$$

$$E_{kp}^{(n)} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-ikx} \int_0^\infty dt\, e^{-pt} E^{(n)}(x, t), \tag{5}$$

it is straightforward to show that for $n$ greater than 1,

$$f_{kp}^{(n)} = \frac{e/m}{p + ikv}[E_{kp}^{(n)} f_0'(v) + S_{kp}^{(n)}], \tag{6}$$

$$E_{kp}^{(n)} = -\frac{4\pi i e^2}{mk}\frac{1}{D_{kp}}\int_C \frac{S_{kp}^{(n)}\, dv}{p + ikv}, \tag{7}$$

$$D_{kp} \equiv 1 - \frac{4\pi i e^2}{mk}\int_C \frac{f_0'(v)\, dv}{p + ikv}, \tag{8}$$

$$S_{kp}^{(n)} \equiv \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} dk' \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dp'}{2\pi i} E_{k-k', p-p'}^{(n-j)} \frac{\partial f_{k'p'}^{(j)}}{\partial v}. \tag{9}$$

In the definition of $D_{kp}$, the contour $C$ is along the real $v$ axis for Re $p > 0$, and passes around the point $v = ip/k$ if Re $p \leq 0$, as in the Landau theory. The contour of $p'$ integration in (9) is such that, on the contour:

(i) Re $(p - p')$ is greater than the real parts of all the singularities of $E_{k-k',p'}^{(n-j)}$, considered as a function of the complex variable $p'$;

(ii) $\sigma$ is a real number, greater than the real parts of all the singularities of $\partial f_{k'p'}^{(j)}/\partial v$, considered as a function of $p'$.

So far, this defines the convolutions in Eq. (9) only for Re $p$ sufficiently large and positive. The possibilities for analytically continuing them in the direction of decreasing Re $p$ (upon which the persistence of damping depends) remain to be discussed.

We restrict ourselves to the "stable" case: $D_{kp} \neq 0$ for Re $p \geq 0$, for any $k \neq 0$.

Since $S^{(n)}$ contains only terms of order less than $n$, and since we know the $n = 1$ solution, we have a formal recipe for generating as many orders as we like in powers of the amplitude. [Clearly, if $f^{(1)}/f_0$ is measured by an amplitude $\epsilon$, then $S^{(n)}$ is $O(\epsilon^n)$.] The process may be viewed as one in which the coupling terms $S^{(n)}$ feed the disturbance into the higher orders as $t$ increases. For more physical distributions (e.g., Maxwellians) the integrals cannot be done explicitly,[5] and we are forced to see what information can be extracted from the purely analytic properties of the $S_{kp}^{(n)}$, considered as a function of the complex variable $p$.

Observe that *if* we can analytically continue $\int S_{kp}^{(n)}\, dv/(p + ikv)$ a finite distance into the left half $p$ plane for all $k$, then the inversion theorem for Laplace transforms,

$$E_k^{(n)}(t > 0) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dp\, e^{pt} E_{kp}^{(n)} \tag{10}$$

tells us that $E_{kp}^{(n)}(t) = O(e^{-\alpha_n t})$ as $t \to \infty$, where $\alpha_n(k)$ is real and positive. This is, of course, subject to integrability requirements on $S_{kp}^{(n)}(v)$; but these appear to be satisfied for all but the most pathological $f_k^{(1)}(v, 0)$ and $f_0(v)$, so we do not labor them here. We assume, to avoid unphysical complications, that:

(a) $f_k^{(1)}(v, 0)$ and $f_0(v)$ are entire functions of $v$;

(b) both these functions and all their derivatives with respect to $v$ are absolutely integrable along lines parallel to the real $v$ axis, and are well behaved and $\to 0$ at infinity there;

(c) $f_k^{(1)}(v, 0)$ is absolutely integrable in $k$, and is well behaved at infinity in $k$.

We assert without proof that these conditions are sufficient to guarantee convergence of all the integrals which appear below.

It will now be shown that $\int S_{kp}^{(2)}\, dv/(p + ikv)$

[5] For distributions which are reciprocals of polynomials, the integrals can be done, at least through second order. See H. B. Liemohn and F. L. Scarf, Phys. Fluids **6**, 490 (1963), Sec. IV.

can be analytically continued a finite distance into the left half of its $p$ plane for all Im $p$. Note that this quantity vanishes for $k = 0$; it is therefore sufficient to show the result for $k \neq 0$. To do this, it is necessary to know where the singularities of $E^{(1)}_{kp}$ and $\partial f^{(1)}_{kp}/\partial v$ are. The singularities of $E^{(1)}_{kp}$, as is well known, are poles which lie on the solutions of

$$D_{kp} = 0. \tag{11}$$

We call the solutions of (11) "$p_i(k)$". By hypothesis, these must all lie in the left half $p$ plane for $k \neq 0$. This is true if $f_0(v)$ is Maxwellian, though the detailed shape of the $p_i(k)$ is not yet known for all $k$.[6] For the following demonstration, no detailed knowledge of $p_i(k)$ is necessary. For ease in visualization, we represent the $p_i(k)$ as in Fig. 1, though more exotic shapes would do equally well. Nor would the following argument be invalidated by possible branches in $p_i(k)$.

We are concerned with the analytic behavior of the integral $\int S^{(2)}_{kp}\, dv/(p + ikv)$, where

$$S^{(2)}_{kp} = \int_{-\infty}^{\infty} dk' \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dp'}{2\pi i} E^{(1)}_{k-k',p-p'} \frac{\partial f^{(1)}_{k'p'}}{\partial v}. \tag{12}$$

For a particular $p$, Re $p > 0$, the singularities of the two terms appearing in the convolution (12) lie on the curves shown in Fig. 2. The plane drawn is the complex $p'$ plane, and the singularities of $\partial f^{(1)}_{k'p'}/\partial v$ lie *somewhere* on the curves opening to the left, while those of $E^{(1)}_{k-k',p-p'}$ lie somewhere on the (displaced) mirror-image curves in the right half-plane. *Any* of the allowed values of $k$, $k'$ can be chosen, and so can any $p$, with Re $p > 0$. The contour of $p'$ integration is shown as a dashed line.

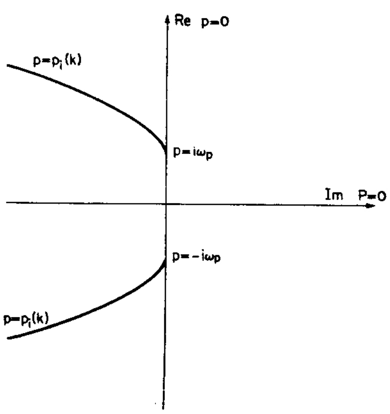


FIG. 1. The solutions of $D_{kp} = 0$ (schematic), with $k$ considered as a variable parameter. As $k \to 0$, $p_i(k) \to \pm i\omega_p$.

[6] In fact, it seems likely that $p_i(k)$ has a doubly infinite number of branches. See J. N. Hayes, Phys. Fluids 4, 1387 (1961), and B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961). It can be shown, however, that for each $k$, there exists a rightmost $p_i(k)$, which lies to the left of the imaginary axis if $k \neq 0$.

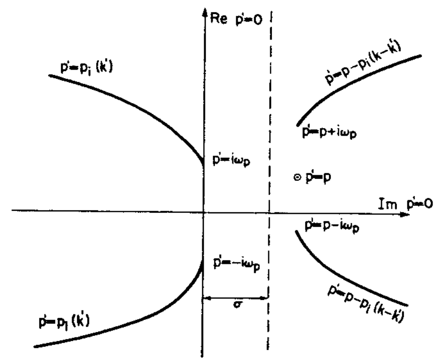


FIG. 2. The complex $p'$ plane for calculating the convolution

$$\int E^{(1)}{}_{k-k',\,p-p'}\, \frac{\partial f^{(1)}{}_{k'p'}}{\partial v}\, dp'$$

(schematic) with Re $p > 0$. The singularities of $E^{(1)}{}_{k-k',p-p'}$ lie on the curves in the right half-plane, those of $\partial f_{k'p'}/\partial v$ on those in the left. The contour of $p'$ integration is shown as a dashed line.

There is also a pole of $\partial f^{(1)}_{k'p'}/\partial v$ at $p' = -ik'v$, but the contour of $v$ integration can be deformed to pass around it in the usual way, upon interchange of $\int dv$ and $\int dp'$, so it cannot contribute a singularity to $E^{(2)}_{kp}$; therefore it is not shown in Fig. 2.

Clearly, the integral is nonsingular for Re $p > 0$. Now allow $p$ to move into its own left half-plane, and observe how a singularity can arise. The contour of $p'$ integration can be deformed to avoid any of the singularities of the integrand (see Fig. 3) until the time when the singularities of $E^{(1)}_{k-k',p-p'}$ and $\partial f^{(1)}_{k'p'}/\partial v$ first meet. Then the singularities can no longer be avoided, and a singularity of the convolution (considered as a function of $p$) can develop.

The first time this can happen is for

$$\begin{aligned} p - p' &= p_i(k - k'), \\ p' &= p_i(k'), \end{aligned} \tag{13}$$

for some $k'$. Eliminating $p'$, and calling the new possible singularities $p_2(k)$,

$$p_2(k) = p_i(k - k') + p_i(k'). \tag{14}$$

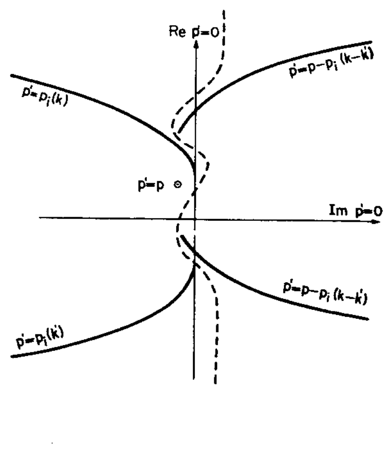


FIG. 3. The contour deformed, for Re $p < 0$. The singularities may be avoided until the singularities of $E^{(1)}{}_{k-k',p-p'}$ "collide" with those of $\partial f^{(1)}{}_{k'p'}/\partial v$, as Re $p$ decreases.

As $k'$ is allowed to range over its permissible values (whichever ones are relevant to the problem at hand), the maximum value of the real part of the right-hand side of (14) is always a finite negative number for $k \neq 0$. In the special case $k = k' = 0$, $E^{(1)}_{k-k',p-p'}$ vanishes and no contribution can arise. Thus $\int S^{(2)}_{kp}(v)\, dv/(p + ikv)$ can *always* be analytically continued a finite distance into its left half $p$ plane.

It will be apparent that an inductive proof to $n$th order involves nothing new, since, as can be seen from (6) and (7), the only properties that need be invoked are that $E^{(1)}_{kp}, \cdots, E^{(n-1)}_{kp}$ be singularity-free for $k \neq 0$, and Re $p >$ some finite negative number which may depend on $k$. It is not so clear what the pictures will look like, and they may become quite tangled. It is important to note that for $E^{(n\geq 2)}_{kp}$, the $\int dv$ must be taken inside all the various convolution integrations, so that there will be no contributions from $p = -ikv$, $p' = -ik'v$, $p'' = -ik''v, \cdots$.

If the $n$th-order singularities [in addition to the $p_i(k)$] are to be called $p_n(k)$, these may arise at

$$p_n(k) = p_{n-j}(k - k') + p_j(k'). \tag{15}$$

Here, $j$ is allowed to run over the numbers $1, 2, \cdots, n - 1$, and $k'$ is to range over all its allowed values. Most generally, these are all the additive combinations of those wave numbers present in the first order, where the sum is understood to contain up to $n - 1$ terms.

Looking back at Eq. (7), it will be seen that there are two types of possible singularities which can contribute to the long-time behavior of $E^{(n)}_k(t)$: the zeros of $D_{kp}$, which are the same in any order; and the singularities of the type of Eq. (15). The $t \to \infty$ form of $E^{(n)}_k(t)$ will be governed by which types have the greater real parts for given $k$. This question will now be discussed.

### III. THE BOUNDED PLASMA

If *all* values of $k$, $0 < |k| < \infty$, are allowed to be present in $f^{(1)}_k(v, 0)$, the question of the "rightmost" singularity of $E^{(n)}_{kp}$ becomes tricky. In fact, it is not possible to bound the damping decrement from below. For example, if one picks $k' = \frac{1}{2}k$ in Eq. (14),

$$\operatorname{Re} p_2(k) = 2 \operatorname{Re} p_i(\tfrac{1}{2}k). \tag{16}$$

Using the usual Landau expression[7] for Re $p_i(k)$ for the Maxwellian case,

$$\frac{\operatorname{Re} p_2(k)}{\operatorname{Re} p_i(k)} \underset{k\to 0}{\longrightarrow} 0. \tag{17}$$

The higher-order poles may move to the right of any particular point in the left half-plane. However, $\int f^{(1)}_k(v, 0)\, dv$ is presumably also going to zero as $k \to 0$, so the question of relative damping rates gets tangled up with the question of the details of the initial wave packet.

The confusion is readily removed by any device which will bound $|k|$ away from zero. Such a device is to confine the plasma[1] by perfectly reflecting walls at $x = 0$ and $x = L$, say.

It is shown in reference 1 that such a plasma is equivalent to a certain unbounded plasma, but with special restrictions. The plasma is defined outside $0 < x < L$ as having a perturbed distribution $f^{(1)}(x, v, 0)$ which is periodic with period $2L$, defined in the region $-L < x < 0$ by

$$f^{(1)}(-x, v, 0) = f^{(1)}(x, -v, 0). \tag{18}$$

It follows at once that

$$f^{(1)}_k(v, 0) = \sum_n \delta(k - k_n) f^{(1)}_{k_n}(v, 0),$$

$$k_n = 0, \pm \pi/L, \pm 2\pi/L, \cdots, \tag{19}$$

$$f^{(1)}_{k_n}(-v, 0) = f^{(1)}_{-k_n}(v, 0), \tag{20}$$

$$E^{(1)}_{kp} = -E^{(1)}_{-k,p}, \qquad k = \text{all } k_n, \tag{21}$$

$$f^{(1)}_{kp}(v) = f^{(1)}_{-k,p}(-v). \tag{22}$$

By virtue of Eq. (3), the $k = 0$ Fourier component has no electric field associated with it. The lowest value of $|k|$ which is present in $E^{(1)}_{kp}$ is, therefore,

$$|k|_{\min} = \pi/L. \tag{23}$$

The point is that the same method-of-images technique applies to the $n$th order; indeed it is easy to show that, given the conditions of (18)–(22) and the formalism of Sec. II,

$$S^{(n)}_{kp}(v) = S^{(n)}_{-k,p}(-v), \qquad n > 1, \tag{24}$$

which satisfies the reflection condition, order by order.[8]

Now consider Eqs. (14) and (15) for the present situation. The values $k$, $k'$ can only be selected from the numbers $0, \pm\pi/L, \pm 2\pi/L, \cdots$, and not more than one of the two can be zero. Generally, $p_i(k = \pm\pi/L)$ is *the* rightmost pole of $E^{(1)}_{kp}$, so it

[7] $$\operatorname{Re} p_i(k) = -\omega_p\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\left(\frac{k_D}{k}\right)^3 \exp\left[-\tfrac{1}{2}\left(\frac{k_D}{k}\right)^2\right].$$
$\omega_p$ is the plasma frequency, $k_D$ is Debye's wavenumber.

[8] Alternatively, one can view this as a result of the nonlinear invariance of Eqs. (1a), (1b) under the transformations $(x, v) \to (x + 2L, v)$; $(x, v) \to (-x, -v)$ for these initial conditions. I am indebted to C. D. Gorman for this remark.

is apparent that the rightmost singularity of $E_{kp}^{(1)}$ can never lie to the right of the rightmost pole of $E_{kp}^{(1)}$. Since $D_{kp} = 0$ provides poles in every order, the $t \to \infty$ behavior, as far as the damping rate goes, is the same for every order.

## IV. DISCUSSION

Insofar as the perfectly reflecting model is valid for laboratory plasmas, and insofar as perturbation theory to all orders is an accurate representation of Eqs. (1), the long-time Landau damping *rate* should show no dependence upon amplitude. As far as we are aware, experimental evidence on this point is totally lacking. It can be made to seem plausible, however, if we note that in the $n$th order, the system to be solved is just the linearized Vlasov equation with the inhomogeneous driving term $S^{(n)}$, which consists of only lower-order, damped terms. Generally, the longest waves damp most slowly, so that in the linear problem, these persist the longest. In the nonlinear problem, however, one is no longer free to specify which $k$ values are present for all time without actually doing a calculation; they are in general quite mixed up by the $S^{(n)}$. The function of the walls is to provide a cutoff in $k$ space below which the disturbance cannot run.

As was the situation at the time of writing reference 4, a satisfactory proof of convergence, divergence, or asymptoticity of the series represented by (6) and (7) is still missing. In view of the complicated form of the $n$th term of the series, it does not appear likely that a proof will be forthcoming in the immediate future. One of the purposes of this calculation was to provide a qualitative prediction by means of this expansion which might either be verified or disproved by a numerical calculation.

In this connection, two points should be made. First, the present theory, like Landau's, says nothing about how long one has to wait before the exponentially damped regime sets in; the result of Sec. III concerns only the magnitude of the eventual damping decrement. Second, when one speaks of the limit $t \to \infty$, one must bear in mind the following limitation, first pointed out by Backus.[9] In the usual linear Landau theory, the ratio which is being neglected is $|E^{(1)}\ \partial f^{(1)}/\partial v|/|E^{(1)} f_0'(v)|$. Due to the presence of undamped terms in the perturbed distribution function which go as exp $(-ikvt)$, this ratio is growing proportionally to $t$. Therefore, eventually, the terms one is throwing away become as large as those one is keeping. If the perturbation is measured by an amplitude $\epsilon$, then clearly this limitation only becomes important after a time of order $1/\epsilon$. Since the Landau damping decrement is independent of $\epsilon$, this does *not* rule out the possibility of seeing Landau damping for long times $\leq O(1/\epsilon)$. It is in this sense that the limit $t \to \infty$ must be understood in the preceding calculation. The difficulty, unfortunately, is present to $n$th order also, and it seems likely that it is an inherent property of Vlasov's equation. A rigorous analysis of the true $t \to \infty$ behavior, of times $\geq O(1/\epsilon)$, might reveal some other type of decay (say, as some power of $1/t$).

The principal alternative to the present expansion which has been put forward so far is the so-called "quasi-linear" theory of electron plasma oscillations.[10] The following are features characteristic of this theory:

(1) The relations

$$f - f_0 = \frac{1}{V^{\frac{1}{2}}} \sum_k \left[f_k e^{i(kx-\omega_k t)} + \left(\begin{smallmatrix}\text{complex}\\ \text{conjugate}\end{smallmatrix}\right)\right],$$

$$E = \frac{1}{V^{\frac{1}{2}}} \sum_k \left[E_k e^{i(kx-\omega_k t)} + \left(\begin{smallmatrix}\text{complex}\\ \text{conjugate}\end{smallmatrix}\right)\right],$$

are assumed valid for all time ($V$ is just a normalization constant).

(2) The $k$th Fourier component of the distribution function is assumed to be given by

$$f_k = \frac{e}{mi} \frac{1}{\omega_k - kv} f_0'(v) E_k$$

for all time.

(3) The $k = 0$ component of $f$, unexpanded, is taken to be the definition of $f_0$.

(4) $f_0$ is assumed to be a "slowly varying" function of time, which develops according to $\partial f_0/\partial t = \partial/\partial v\ (D\ \partial f_0/\partial v)$ where

$$D = e^2/m^2\, 2\pi \sum_k |E_k|^2\, \delta(\omega_k - kv).$$

(5) $\omega_k$ is determined as a "slowly varying" function of time by the $t \to \infty$ Landau relation.

We must confess a lack of understanding of the motivations, both physical and mathematical, of these assumptions (none is awarded a proof). Assumptions (1) and (2), particularly, seem to run contrary to the entire body of results of the linear theory, in which the equation for the electric field is satisfied only in the $t \to \infty$ limit, and in which the relation for the distribution function is never satisfied, except for a highly singular (and therefore

[9] G. Backus, J. Math. Phys. 1, 178 (1960). See Sec. VIII.

[10] A. A. Vedenov, E. P. Velikhov, R. Z. Sagdeev, Nucl. Fusion 1, 82 (1961); *ibid.*, 1962 Suppl., p. 465. A somewhat similar theory dealing with the unstable case was put forward at the 1961 Salzburg Conference by W. E. Drummond and D. Pines, and will also appear in a Supplement to Nuclear Fusion.

physically unlikely) set of initial conditions.[11] The entire theory seems to be based on a (tacit) attempt to force too detailed an analogy between the behavior of a collisionless plasma and that of a fluid-type continuous medium. Landau showed unequivocally that for finite times, *all* frequencies are present in the spectrum of the $k$th Fourier component of the electric field, for an arbitrary initial perturbation. Any attempt to identify "time scales" on which the various quantities evolve will call for a high level of mathematical precision, indeed.

*Note added in proof.* Given the slab conditions of Sec. III, a demonstration nearly identical to that of Sec. II shows that, not only is $E_{kp}^{(n)}$ singularity-free in the region $\operatorname{Re} p > 0$, but $f_{kp}^{(n)}$ is also without singularities in this region, except for a simple pole at $p = -ikv$. This shows explicitly that this expansion is not plagued by the "secular terms" which accompany straightforward expansions in powers of the particle interactions. [See: E. A. Frieman, J. Math. Phys. **4**, 410 (1963), and E. A. Frieman, S. Bodner, and P. Rutherford, Princeton Plasma Physics Lab. Report MATT-169 (unpublished).]

The $kt$ terms of Backus, described in Sec. IV of this present work, are of a qualitatively different character from the usual "secular terms," and have not, to our knowledge, been successfully handled by anyone as yet. They are quite generally present for both the bounded and unbounded cases, and are not contingent upon any special properties of the $D_{kp}$ function. The proof of the absence of the ordinary "secular terms" applies only in case there is a lower bound in $k$ space.

## ACKNOWLEDGMENTS

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## APPENDIX: EQUIVALENCE OF THE *N*TH ORDER NORMAL MODE AND LAPLACE TRANSFORM SOLUTIONS

We include a demonstration of the equivalence of the two methods of doing perturbation theory to all orders given in reference 4—Laplace transform and singular normal mode. This is offered not so much because it fits logically with the preceding material, but because that it is felt that it remedies a lack of reference 4.

To save space, we refer to reference 4 for notation. The problem is to show that by substituting the "initial value" of Eq. (41) of that paper,

$$\text{“}f_k^{(n)}(v,0)\text{”} = \frac{e}{mi} P \int \frac{S_{k,kv}^{(n)}(v') + S_{k,kv'}^{(n)}(v)}{v - v'}\, dv' \tag{25}$$

into the van Kampen–Case[11] expression for $E_k^{(1)}(t > 0)$ as a functional of $f_k^{(1)}(v, 0)$, one arrives at the same expression that one gets from Eqs. (7) and (10) of this communication. In (25), the $S_{k\omega}^{(n)}$ which appears is a *Fourier* transform, defined by $2\pi S_{k\omega}^{(n)} = S_{kp}^{(n)}$, if $p = -i\omega$.

For the van Kampen–Case $E_k^{(1)}(t > 0)$, one has

$$E_k^{(1)}(t > 0) = \frac{2e}{k} \int_{-\infty}^{\infty} A_+(\nu) e^{-ik\nu t}\, d\nu, \tag{26}$$

$$A_+(\nu) = \lim_{\epsilon\to 0+} \left[ \int \frac{f_k^{(1)}(v,0)\, dv}{v - \nu - i\epsilon} \right] \Big/ \left[ 1 - \frac{4\pi e^2}{mk^2} \cdot \int \frac{f_0'(v)\, dv}{v - \nu - i\epsilon} \right]. \tag{27}$$

Replacing $f_k^{(1)}(v, 0)$ by Eq. (25), and using the Plemelj formula for the principal value function,

$$-\frac{e}{mi} P \int \frac{S_{k,k\nu}^{(n)}(v') + S_{k,kv'}^{(n)}(v)}{v - v'}\, dv' = \frac{e}{mi} \int \frac{S_{k,k\nu}^{(n)}(v') + S_{k,kv'}^{(n)}(v)}{v' - v - i\epsilon}\, dv' - 2\pi i \frac{e}{mi} S_{k,kv}^{(n)}(v), \tag{28}$$

where we have used the analyticity of $S_{k\omega}^{(n)}$ for $\operatorname{Im} \omega > 0$.

Noting that as $\epsilon \to 0_+$,

$$\int \frac{dv' S_{k,kv'}^{(n)}(v'')}{(v' - \nu - i\epsilon)(v'' - v' - i\epsilon)} = 2\pi i \frac{S_{k,k\nu}^{(n)}(v'')}{v'' - \nu - i\epsilon},$$

$$\int \frac{dv'' S_{k,kv''}^{(n)}(v')}{v'' - v' - i\epsilon} = 2\pi i\, S_{k,kv'}^{(n)}(v'),$$

we have that

$$\int \frac{dv'}{v' - \nu - i\epsilon} \text{“}f_k^{(n)}(v,0)\text{”} = \frac{2\pi e}{m} \int \frac{dv''\, S_{k,k\nu}^{(n)}(v'')}{v'' - \nu - i\epsilon}. \tag{29}$$

Substituting into (26) and (27) gives:

$$E_k^{(n)}(t > 0) = \frac{2e}{k} \int_{-\infty}^{\infty} d\nu e^{-ik\nu t} \frac{\dfrac{2\pi e}{m} \displaystyle\int \frac{dv''\, S_{k,k\nu}^{(n)}(v'')}{v'' - \nu - i\epsilon}}{1 - \dfrac{4\pi e^2}{mk^2} \displaystyle\int \frac{f_0'(v)\, dv}{v - \nu - i\epsilon}}. \tag{30}$$

This is just what one gets from Eqs. (7) and (10).

[11] N. G. van Kampen, Physica **21**, 949 (1955). K. M. Case, Ann. Phys. (N. Y.) **7**, 349 (1959).